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A Review on Some Classes of Metric Spaces Lying between the Classes of Compact and Complete Metric Spaces

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Abstract

In the theory of classical analysis, metric properties which are intermediate between compactness and completeness have turned out to be an interesting area of study in the last six decades. In this article, we discuss the advancement by reviewing certain classes of metric spaces satisfying such properties.
Keywords: Atsuji spaces, Boundedly UC metric spaces, 'Almost' UC metric spaces, Bourbaki complete metric spaces, Cofinally complete metric spaces, Cofinally Bourbaki complete metric spaces.

1. Introduction

The concepts of compactness and completeness are central to the study of metric spaces. Every compact metric space is complete. The converse, however, may not be true. As a counterexample we may consider the euclidean space $\mathbb{R}^n$ where n is a positive integer. The class of compact metric spaces, thus, forms a proper subcollection of the class of complete metric spaces.

For more than six decades it remains a common interest for topologists to find new classes of metric spaces that are intermediate between these two classes. It should be noted that searching for such a class actually boils down to the question of finding metric property that demands a larger class than the class of Cauchy sequences to be convergent. Probably J. Nagata [16] was the first to make an attempt in finding such an intermediate class of spaces. His study was later extended by A. A. Monteiro and M. M. Peixoto [13] in 1951. However it was Atsuji [1] who first made a systematic study on this newly introduced metric space, which is now widely known as Atsuji space or UC metric space.

Apart from UC metric space, subsequently other kinds of metric spaces were also introduced that reside in such an intermediate class. Boundedly UC metric space [6], cofinally complete metric space [2], strongly cofinally complete metric space [3] are among the most important ones. Recently in 2014, Bourbaki complete metric space [10], cofinally Bourbaki complete metric space [10] and, in 2017, 'almost' UC metric space or AUC metric space [11] were introduced. Those spaces are not only interesting in their own rights but also helpful in answering questions arising from Convex Analysis, Optimization Theory and from the setting of Convergence Structures on Hyperspaces. For more details readers are advised to consult [7] and the references therein.

The general aim of the present article is to introduce the concept of those spaces and explore various facets of the related metrics; a comparative study between such spaces follows as a natural consequence.

In order to discuss a large number of results involving various metric spaces under discussion, the present study starts with the recollection of the necessary facts from the topology of metric spaces in Section 2, most of which are well-known. Subsequently, in the following six sections six

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undermentioned metric spaces have been introduced and explored, each of which is complete but fails to be compact in general.

Section 3: UC metric spaces or Atsuji spaces
Section 4: Boundedly UC metric spaces
Section 5: 'Almost' UC metric spaces or AUC metric spaces
Section 6: Bourbaki complete metric spaces
Section 7: Cofinally complete metric spaces
Section 8: Cofinally Bourbaki complete metric spaces

These spaces have been developed over a span of 60 years starting from the early fifties of the last century (UC space or Atsuji space), the latest ('Almost' UC space or AUC space) being introduced in 2017. In order to have easier understanding of their relationships and to identify the major theme of the topic, their chronological orders are often violated and they are put together in a list that best serves the purpose. The same saying goes on for the results and ideas related to a particular section, as well. They are arranged, introduced or eliminated according to the need of the study.

Throughout the article, $\mathbb{R}$ stands for the set of real numbers and $\mathbb{N}$, the set of natural numbers. Unless otherwise stated, $\mathbb{R}$ is endowed with the usual distance metric and X,Y are metric spaces. If (X,d) is a metric space then for $c \in X$, $B(c,\delta)$ denotes the open ball of radius δ centered at c and for $A \subset X$, $\text{diam}(A)$ denotes its diameter measured as $\inf\{d(x,y) : x,y \in A\}$, $\overline{A}$ denotes the closure of A, A^d denotes the set of all accumulation points of A. For every pair of nonempty disjoint subsets A and B of (X,d), $d(A,B)$ denotes the distance between A and B.

2. Preliminaries

Definition 1. Let $(X,d),(Y,d')$ be metric spaces and $f : X \to Y$ be a function.

f is said to be continuous on X if for $c \in X, \varepsilon > 0$ there exists $\delta > 0$ such that $\forall\, x \in X$ with $d(x,c) < \delta$ we have $d'(f(x), f(c)) < \varepsilon$.

f is said to be uniformly continuous on X if for $\varepsilon > 0$ there exists $\delta > 0$ such that $\forall\, x,y \in X$ with $d(x,y) < \delta$ we have $d'(f(x), f(y)) < \varepsilon$.

Any uniformly continuous function between metric spaces is continuous. The converse, however, may not be true.

Theorem 1. Let $(X,d),(Y,d')$ be metric spaces and $f : X \to Y$ be a function. Then the following conditions are equivalent:

a) f is uniformly continuous.

b) For every pair of nonempty disjoint subsets A,B of X,

$$d(A,B) = 0 \implies d'(f(A), f(B)) = 0.$$

Definition 2. A metric space X (resp. a subset A of X) is said to be compact if every open cover of X (resp. A) has a finite subcover.

Theorem 2. Let X be a metric space and $A \subset X$. Then the following conditions are equivalent:

a) A is a compact set.

b) Every sequence in A has a subsequence that converges to some point of A.

c) Every infinite subset of A has an accumulation point in A.

Definition 3. Let X be a metric space and $\mathscr{U}$, an open cover of X. A positive number δ is called a Lebesgue number of $\mathscr{U}$ if for each $A \subset X$ with $\text{diam}(A) < \delta$, $A \subset U$ for some $U \in \mathscr{U}$.

X is said to be Lebesgue if every open cover of X has a Lebesgue number.

Theorem 3. Let (X,d) be a compact metric space. Then

a) Every open cover of X has a Lebesgue number.

b) For every pair of nonempty closed disjoint subsets A,B of X, $d(A,B) > 0$.

c) Every continuous function on X is uniformly continuous.

Definition 4. Let (X,d) be a metric space.

A sequence (x_n) in a metric space X is said to be Cauchy if for each $\varepsilon > 0$ $\exists$ $k \in \mathbb{N}$ such that $d(x_m,x_n) < \varepsilon$, $\forall$ $m,n \geq k$.

X is said to be complete if every Cauchy sequence in X converges.

Definition 5. Let (X,d) be a metric space. A subset A of X is called an ε-net if $X = \bigcup_{a\in A} B(a,\varepsilon)$.

X is said to be totally bounded if for each $\varepsilon > 0$, X has a finite ε-net.

Theorem 4. A metric space is compact if and only if it is complete and totally bounded.

For relevant notions on uniform spaces readers may consult [15].

It is known that every metric structure on a set generates a uniform structure on it and every uniformity on a set generates a topology on it.

Definition 6. Let $(X,\mathscr{U}),(Y,\mathscr{U})$ be uniform spaces and $f : X \to Y$ be a function.

f is called continuous on X if it is continuous with respect to the topologies generated by $\mathscr{U}$ and $\mathscr{U}'$ on X and Y respectively.

f is called uniformly continuous on X if for $V \in \mathscr{U}'$ there exists $U \in \mathscr{U}$ such that $\forall$ $x,y \in X$ with $(x,y) \in U$, we have $(f(x),f(y)) \in V$.

It can be shown that for two metric spaces (X,d), (Y,d') and a map $f : X \to Y$, f is uniformly continuous if and only if $f : (X,\mathscr{U}_d) \to (Y,\mathscr{U}_{d'})$ is uniformly continuous where $\mathscr{U}_d$ and $\mathscr{U}_{d'}$ are uniformities generated by d and d' respectively.

3. UC metric spaces or Atsuji spaces

Though for a compact metric space X (a) any continuous function from X to a metric space Y is uniformly continuous, (b) for any pair of nonempty disjoint, closed subsets A,B of X, we have $d(A,B) > 0$, and (c) each open cover of X has a Lebesgue number, it is important to note that, none of those properties characterizes compactness of a metric space. They are, in fact, the characterizing properties of a bigger class of metric spaces known as UC metric spaces or Atsuji spaces.

Definition 7. [1] A metric space (X,d) is called an *UC metric space* or *Atsuji space* if every real-valued continuous function on X is uniformly continuous.

Theorem 5. [14] An UC metric space is complete.

Remark 1. A complete metric space may not be UC as is evident from the case of $\mathbb{R}$. Thus the class of UC metric spaces forms a proper subset of the class of complete metric spaces. On the other hand, any real-valued continuous map defined on a compact metric space is uniformly continuous. The converse, however, may fail. An infinite discrete metric space may be taken as a counterexample. Thus we arrive at the following inclusion relation: the class of compact metric spaces $\subsetneq$ the class of UC metric spaces $\subsetneq$ the class of complete metric spaces.

The definition of UC metric spaces can be extended for continuous functions defined from X to any metric space, instead of just $\mathbb{R}$. The range of the function, in fact, can be extended to any uniform space by considering the uniform structure of the metric space X. On the other hand, the same definition can be realized by considering only the bounded real-valued continuous functions on X. Thus the following result can be established.

Theorem 6. ([1], [12], [18]) Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is an UC metric space.

b) For an arbitrary metric space Y and a continuous function $f : X \to Y$, f is uniformly continuous.

c) For an arbitrary uniform space Y and a continuous function $f : X \to Y$, f is uniformly continuous.

d) Every bounded real-valued continuous function on X is uniformly continuous.

It is known that the class of Lebesgue metric spaces $\mathscr{L}$ is strictly larger than the class of compact metric spaces. The class $\mathscr{L}$, in fact, coincides with the class of UC metric spaces.

Theorem 7. [13] Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is an UC metric space.

b) X is a Lebesgue metric space.

c) Every open cover of X by two sets has a Lebesgue number.

It has already been noted that existence of a positive distance between any pair of disjoint closed subsets is the characterizing property of UC metric spaces. In fact, more can be said by involving the idea of closures of subsets.

Theorem 8. ([1], [18]) Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is an UC metric space.

b) Let $(A_n)_n$ be a sequence of subsets of X such that $\bigcap\limits_{n=1}^{\infty} \overline{A_n} = \emptyset$. Then $\bigcap\limits_{n=1}^{\infty} B(A_n, r) = \emptyset$ for some $r > 0$.

c) Let A_1, A_2 be subsets of X such that $\overline{A_1} \bigcap \overline{A_2} = \emptyset$. Then $B(A_1, r) \bigcap B(A_2, r) = \emptyset$ for some $r > 0$.

d) Let A_1, A_2 be non-empty closed subsets of X such that $A_1 \bigcap A_2 = \emptyset$. Then $d(A_1, A_2) > 0$.

Much alike the case of complete and compact metric spaces, UC metric spaces can also be characterized in terms of *sequences*. In order to state the related result, we require the following definitions.

Definition 8. Let (X,d) be a metric space.

Two sequences (x_n) and (y_n) in X are called *asymptotic*, in notation $(x_n) \asymp (y_n)$, if for $\varepsilon > 0\ \exists\, k \in \mathbb{N}$ such that $d(x_n, y_n) < \varepsilon\ \forall\, n \geq k$. For example, $(n) \asymp \left(n + \frac{1}{n}\right)$ in $\mathbb{R}$.

A sequence of distinct isolated points (x_n) in X is said to be a sequence of *paired isolated points* if $\lim\limits_{n\to\infty} d(x_{2n-1}, x_{2n}) = 0$. For example, consider $\left(\frac{1}{n}\right)$ in $\mathbb{R}$.

A sequence (x_n) in X is called *pseudo-Cauchy* if for $\varepsilon > 0$ and $k \in \mathbb{N}$, $\exists\, p, q \geq k$ with $p \neq q$ such that $d(x_p, x_q) < \varepsilon$. Clearly in a pseudo-Cauchy sequence two terms get arbitrarily close frequently rather than being eventually.

Theorem 9. ([4], [5], [17], [18]) Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is an UC metric space.

b) If $(x_n) \asymp (y_n)$ in X where $x_n \neq y_n\ \forall\, n \in \mathbb{N}$, then (x_n) (equivalently (y_n)) has a cluster point in X.

c) Every sequence of paired isolated points in X has a convergent subsequence and every sequence in X^d has a convergent subsequence.

d) Every pseudo-Cauchy sequence of distinct points in X has a convergent subsequence.

We finish this section with the characterizations of UC metric spaces in terms of *nested sequences of closed subsets* (of the underlying metric spaces) and *finite intersection property*.

Definition 9. For a subset A of a metric space X, set

i) $\alpha(A) = \inf\{\varepsilon > 0 : A$ can be covered by a finite number of balls in X with diameter less that $\varepsilon\}$ if $A \neq \emptyset$ and $\alpha(\emptyset) = 0$. Here α is called the *Kuratowski measure of noncompactness*.

ii) $\chi(A) = \inf\{\varepsilon > 0 : A$ can be covered by $\bigcup_{x\in F} B(x,\varepsilon)$ for some finite set $F \subset X\}$ if $A \neq \emptyset$ and $\chi(\emptyset) = 0$. Here χ is called the *Hausdroff measure of noncompactness*.

iii) $\overline{d}(A) = \sup\{d(a, X\{a\}) : a \in A\}$.

iv) $\underline{d}(A) = \sup\{d(a, X\{a\}) : a \in A\}$.

We recall, at this stage, that completeness of a metric space X is equivalent to any of the following conditions:

(a) For a nested sequence (F_n) of nonempty closed sets of X with $\lim_{n\to\infty} \alpha(F_n) = 0$, $\bigcap_{n=1}^{\infty} F_n$ is a nonempty compact set.

(b) For a nested sequence (F_n) of nonempty closed sets of X with $\lim_{n\to\infty} \chi(F_n) = 0$, $\bigcap_{n=1}^{\infty} F_n$ is a nonempty compact set.

Analogous characterizations for UC metric spaces can be obtained as follows:

Theorem 10. [5] Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is an UC metric space.

b) For a nested sequence (F_n) of nonempty closed sets of X with $\lim_{n\to\infty} \overline{d}(F_n) = 0$, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

c) For a nested sequence (F_n) of nonempty closed sets of X with $\lim_{n\to\infty} \underline{d}(F_n) = 0$, $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

d) Consider a family $\mathscr{A}$ of closed subsets of X with the finite intersection property. Then for a sequence (A_n) in $\mathscr{A}$ with $\lim_{n\to\infty} \overline{d}(A_n) = 0$, $\bigcap \mathscr{A} \neq \emptyset$.

4. Boundedly UC metric spaces

A metric space is boundedly compact if all of its closed and bounded subsets are compact. Recall for metric spaces, Cauchy sequences having convergent subsequences are convergent on their own, Thus every boundedly compact metric space is complete. It is important to note that there exists a class of metric spaces that plays the same role corresponding to the class of boundedly compact metric spaces as the class of UC metric spaces plays with respect to the class of compact metric spaces. These metric spaces are known as boundedly UC metric space.

Definition 10. [6] A metric space X is called *boundedly UC metric space* if every closed and bounded subset of X is an UC metric space.

Euclidean spaces $\mathbb{R}^n$ are clearly boundedly UC metric space.

Similar to the case for UC metric spaces, boundedly UC metric spaces can be characterized in terms of uniform continuity of continuous functions. It is evident from the following result.

Theorem 11. [6] Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is a boundedly UC metric space.

b) For an arbitrary metric space Y and a continuous function $f : X \to Y$, f is uniformly continuous on non-empty bounded subsets of X.

c) For an arbitrary uniform space Y and a continuous function $f : X \to Y$, f is uniformly continuous on non-empty bounded subsets of X.

d) Every bounded real-valued continuous function on X is uniformly continuous on non-empty bounded subsets of X.

Remark 2. Cosider two metric spaces (X,d) and (Y,d'), of which X is boundedly UC. Let $f : X \to Y$ be continuous. It follows from the last result that for any non-empty bounded subset B of X and $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\forall\, x,y \in B,\ d(x,y) < \delta \implies d'(f(x),f(y)) < \varepsilon.$$

This property clearly holds for any non-empty compact subset B of a metric space.

We know that in a UC metric space two disjoint closed subsets are positive distance apart. An analogous result can be obtined for boundedly UC metric spaces:

Theorem 12. [6] Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is a boundedly UC metric space,

b) Let A_1,A_2 be non-empty closed and bounded subsets of X such that $A_1 \bigcap A_2 = \emptyset$. Then $d(A_1,A_2) > 0$,

b) Let A_1,A_2 be non-empty closed subsets of X of which A_1 is bounded such that $A_1 \bigcap A_2 = \emptyset$. Then $d(A_1,A_2) > 0$.

There exists a Lebesgue like criterion for boundedly UC metric spaces.

Theorem 13. ([6], [9]) Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is a boundedly UC metric space.

b) For an open cover $\mathscr{V} = \{V_i : i \in I\}$ of a closed and bounded subset B of X, $\exists\ \delta > 0$ such that for any subset A of X with diam$(A) < \delta$,

$$A \cap B \neq \emptyset \implies A \subset V,$$

for some $V \in \mathscr{V}$.

We finish this section with a sequencial criterion for boundedly UC metric spaces.

Theorem 14. ([6], [9]) Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is a boundedly UC metric space.

b) All closed and bounded subsets of X^d are compact and all bounded paired sequence of isolated points have cluster points.

c) If (x_n) is a bounded sequence with $\lim\limits_{n\to\infty} d\left(x_n, \{x_n\}^c\right) = 0$, then (x_n) has a cluster point.

5. 'Almost' UC metric spaces or AUC metric spaces

Definition 11. [11] Let X and Y be two metric spaces.

A function $f : X \to Y$ is said to be *almost bounded* on X if for chosen $\varepsilon > 0$ there is a $\delta > 0$ such that the following holds: corresponding to a subset A of X with diam$(A) < \delta$, there is a finite subset C of Y such that $f(A) \subset \bigcup\limits_{c \in C} B(c,\varepsilon)$.

A continuous function $f : X \to Y$ is said to be *almost uniformly continuous* if it is almost bounded on X.

X is said to be *'Almost' UC* or *AUC* if every continuous real-valued function on X is almost uniformly continuous.

For a function $f : X \to Y$ it is straightforward to see that,
a) If f is uniformly continuous then f is almost uniformly continuous,
b) If $f(X)$ is totally bounded then f is almost bounded.
Moreover,

Theorem 15. [11] If every closed and bounded subset of X is compact, then every continuous function from a closed subspace of X to Y is almost uniformly continuous.

Thus there are functions which are unbounded, almost uniformly continuous but not uniformly continuous. On the other hand, $f(x) = \frac{1}{x},\ x \in (0,\infty)$ is an example of continuous, not almost uniformly continuous function [11].

Theorem 16. [11] Every AUC metric space is complete.

It is well-known [15] that a metric space is compact iff it is complete and totally bounded. Thus, in view of the above discussion we may conclude the following:

Corollary 1. [11] A metric space is compact iff it is totally bounded and AUC.

It is clear from the above results that the class of AUC metric spaces is intermediate between the classes of complete and UC metric spaces. However those classes are all distinct as we will see.

Definition 12. [11] Let X be a metric space and $\mathscr{U}$, an open cover of X. A positive number $\delta > 0$ is said to be a *weak Lebesgue number* of $\mathscr{U}$ if for every $A \subset X$ with $\mathrm{diam}(A) < \delta$, there exists a finite subset $\mathscr{U}'$ of $\mathscr{U}$ such that $A \subset \bigcup \mathscr{U}'$.

X is said to be *weakly Lebesgue* (resp. *weakly countably Lebesgue*) if every open cover (resp. countable open cover) of X has a weak Lebesgue number.

Theorem 17. [11] For a metric space X, the following are equivalent:
a) X is AUC.
b) X is weakly Lebesgue.
c) X is weakly countably Lebesgue.
d) A continuous function $f : X \to Y$ (Y being any metric space) is almost uniformly continuous.

Example 1. [11] (*A weakly Lebesgue, non-Lebesgue metric space*) Consider, for each $n \in \mathbb{N}$, a finite set A_n having at least two points such that $A_m \bigcap A_n = \emptyset,\ \forall\, m \neq n$. Set $X = \bigcup_{n\in\mathbb{N}} A_n$. We define $d : X \times X \to \mathbb{R}$ by:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } x,y \in A_n, x \neq y \\ 1 & \text{if } x \in A_m, y \in A_n, m \neq n \end{cases}$$

Then (X,d) forms a discrete metric space. We note that $\{\{x\} : x \in X\}$ is an open cover of X without any Lebesgue number. Thus X is not Lebesgue. However given an open cover $\mathscr{U}$ of X, any subset of X having diameter less than 0.5 is contained in some A_n. Thus A can be put inside a finite union of members of $\mathscr{U}$. So X is weakly Lebesgue. □

Recall from [13] that a metric space is Lebesgue iff it is UC. Thus we see that the class of AUC metric spaces is strictly larger than the class of UC metric spaces.

Example 2. [11] Let $X = \mathbb{N} \times \mathbb{N}$ and $A_n = \{n\} \times \mathbb{N}$ for each $n \in \mathbb{N}$. We define $d : X \times X \to \mathbb{R}$ by:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \frac{1}{n} & \text{if } x,y \in A_n, x \neq y \\ 1 & \text{if } x \in A_m, y \in A_n, m \neq n \end{cases}$$

Then (X,d) is complete but fails to be an AUC metric space. Consequently, the class of compact metric spaces $\subsetneq$ the class of UC metric spaces $\subsetneq$ the class of AUC metric spaces $\subsetneq$ the class of complete metric spaces.

6. Bourbaki complete metric spaces

Any metric property stronger than completeness and weaker than compactness demands the sequences of a larger class than that of Cauchy sequences to cluster. In what follows, we revisit Bourbaki-Cauchy sequences, collection of which forms such a class.

It is known that total boundedness can be characterized in terms of Cauchy sequences (*A metric space is totally bounded iff every sequence has a Cauchy subsequence*). In this section, we discuss a new notion of boundedness, to be called Bourbaki-boundedness. Bourbaki-boundedness can be characterized using Bourbaki-Cauchy sequences and an intermediate class between that of compact and complete metric spaces can be constructed by demanding all Bourbaki-Cauchy sequences to cluster.

Definition 13. [10] Let (X,d) be a metric space and $A \subset X$.

For $\varepsilon > 0$, the *ε-enlargement of A* is given by $A^{\varepsilon} = \bigcup\limits_{a \in A} B(a,\varepsilon) = \{y \in X : d(y,A) < \varepsilon\}$.

For $\varepsilon > 0$ and $n \in \mathbb{N}$, the *n^{th} ε-enlargement of the ball $B(x,\varepsilon)$*, denoted by $B^n_{\varepsilon}(x)$, is defined as follows: $B^1_{\varepsilon}(x) = B(x,\varepsilon)$ and $B^n_{\varepsilon}(x) = (B^{n-1}_{\varepsilon}(x))^{\varepsilon}$, $\forall\, n \geq 2$.

A subset F of X is called *Bourbaki-bounded* if for $\varepsilon > 0$ $\exists\, m \in \mathbb{N}$ and finitely many points $p_1, p_2, \cdots, p_k \in X$ such that $F \subset \bigcup\limits_{i=1}^{k} B^m_{\varepsilon}(p_i)$.

Clearly for $m = 1$, F becomes a totally bounded subset of X. Thus a totally bounded subset is necessarily Bourbaki-bounded. On the other hand, every Bourbaki-bounded subset of X is a bounded subset.

We note that the collection $\mathscr{B}$ of Bourbaki-bounded subsets of X forms a *bornology* in X: (a) $\{x\} \in \mathscr{B}$, $\forall\, x \in X$, (b) $B \in \mathscr{B}, A \subset B \implies A \in \mathscr{B}$, (c) $A,B \in \mathscr{B} \implies A \bigcup B \in \mathscr{B}$.

Before introducing the next result, we note that a subset of a metric space X can be Bourbaki-bounded in X but not in itself. Consider the following example.

Example 3. [10] Let $l_2 = \{(x_n) : (x_n)$ is a real sequence and $\sum\limits_{n=1}^{\infty} |x_n|^2 < \infty\}$ be endowed with the 2-norm and X denotes the metric space induced by l_2. If $B = \{e_n : n \in \mathbb{N}\}$ represents the standard basis of l_2, then B is a Bourbaki-bounded subset of X that fails to be Bourbaki-bounded in itself.

Theorem 18. ([1], [8]) Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is Bourbaki-bounded subset of itself.

b) For every uniformly continuous function $f : X \to \mathbb{R}$, $f(X)$ is bounded in $\mathbb{R}$.

Definition 14. [10] A sequence (x_n) in a metric space (X,d) is called *Bourbaki-Cauchy* if for $\varepsilon > 0$ $\exists\, m,k \in \mathbb{N}$, $p \in X$ such that $x_n \in B^m_{\varepsilon}(p)$ $\forall\, n \geq k$.

Note that every Cauchy sequence is Bourbaki-Cauchy. However the converse may not be true.

Recall that a subset B of a metric space X is totally bounded if and only if every sequence in B has a Cauchy subsequence [19]. An analogous result can be obtained for Bourbaki-bounded subsets.

Theorem 19. [10] Let (X,d) be a metric space and $B \subset X$. Then the following conditions are equivalent:

a) B is a Bourbaki-bounded subset of X.

b) Every countable subset of B is a Bourbaki-bounded subset of X.

c) Every sequence in B has a Bourbaki-Cauchy subsequence in X.

Definition 15. [10] A metric space X is called *Bourbaki complete* if every Bourbaki-Cauchy sequence in X has a convergent subsequence.

Every Bourbaki complete metric space is complete. The converse may not be true as is evident from the last example: (e_n) is a Bourbaki-Cauchy sequence without any convergent subsequence. Recall that a compact metric space is sequentially compact. Thus compact metric spaces are Bourbaki complete. Consequently, the class of Bourbaki complete metric spaces lies in between the classes of compact and complete metric spaces.

It is well-known that compact metric spaces can characterized in terms of totally boundedness and completeness (*A metric space is compact if and only if it is complete and totally bounded*). A similar result can be obtained by replacing completeness with Bourbaki-completeness and totally boundedness with Bourbaki-boundedness.

Theorem 20. [10] A metric space is compact if and only if it is Bourbaki-bounded and Bourbaki-complete.

Another characterization of compactness in terms of Bourbaki-completeness is as follows:

Theorem 21. [10] A metric space is Bourbaki complete if and only if closure of every Bourbaki-bounded subset is compact.

A complete metric space can be characterized by Kuratowski measure of noncompactness and Hausdroff measure of noncompactness. Similar results can be obtained for Bourbaki-complete metric spaces by generalizing the above measures.

Definition 16. Let X be a metric space and $A\ (\neq \emptyset) \subset X$.Define

$\gamma(A) = \inf\{\varepsilon > 0 : A \subset B^m_\varepsilon(x)$ for some $m \in \mathbb{N}, x \in X\}$, and

$\eta(A) = \inf\{\varepsilon > 0 : A \subset \bigcup_{i=1}^{k} B^m_\varepsilon(x_i)$ for some $m \in \mathbb{N}$ and finitely many $x_1, x_2, \cdots, x_k \in X\}$.

Clearly $\eta(A)$ can be thought of as a generalization of Hausdroff measure of noncompactness.

Theorem 22. [10] Let (X,d) be a metric space. Then the following conditions are equivalent:

a) X is Bourbaki-complete.

b) For a nested sequence (F_n) of nonempty closed sets of X with $\lim_{n\to\infty} \eta(F_n) = 0$, $\bigcap_{n=1}^{\infty} F_n$ is a nonempty compact set.

c) For a nested sequence (F_n) of nonempty closed sets of X with $\lim_{n\to\infty} \gamma(F_n) = 0$, $\bigcap_{n=1}^{\infty} F_n$ is a nonempty compact set.

7. Cofinally complete metric spaces

Definition 17. [2] A sequence (x_n) in a metric space (X,d) is *cofinally Cauchy* if for $\varepsilon > 0$ $\exists$ an infinite subset $\mathbb{N}_\varepsilon$ of $\mathbb{N}$ such that $d(x_m,x_n) < \ \forall\, m,n \in \mathbb{N}_\varepsilon$.

A metric space X is called *cofinally complete* if every cofinally Cauchy sequence in X has some convergent subsequence.

It is to be noted that the elements in a cofinally Cauchy sequence are arbitrarily close to each other frequently, rather than eventually.

Since every Cauchy sequence is cofinally Cauchy, a cofinally complete metric space is complete. The converse, however, may fail. It is evident from the following proposition.

Proposition 1. [2] A Banach space (i.e. a complete normed linear space) is cofinally complete if and only if it is of finite dimension.

On the other hand, a compact metric space, being sequentially compact, is cofinally complete. Thus, the class of cofinally complete metric spaces lies in between the classes of compact and complete metric spaces.

Example 4. [2] $\mathbb{R}$ is cofinally complete.

A comparison between cofinally complete metric spaces and UC metric spaces can be made using the following proposition.

Proposition 2. [2] Let (x_n) be a cofinally Cauchy sequence without any constant subsequence in a metric space (X,d). Then there is a pairwise disjoint family $\{\mathscr{M}_j\}_{j\in\mathbb{N}}$ of infinite subsets of $\mathbb{N}$ such that

a) if $p,\ q\ (p \neq q) \in \bigcup\limits_{j\in\mathbb{N}} \mathscr{M}_j$, then $x_p \neq x_q$,

b) if $p,\ q \in \mathscr{M}_m$, then $d(x_p,x_q) < \frac{1}{m}$.

Consider a cofinally Cauchy sequence (x_n) in a UC metric space X. If (x_n) has a constant subsequence, then (x_n) clusters. Otherwise, in view of the last proposition, (x_n) has some pseudo-Cauchy subsequence of distinct terms. Since X is UC, (x_n) has a convergent subsequence. Thus every UC metric space is cofinally complete. However the converse may not be true as is evident from the last example.

Definition 18. [2] Let (X,d) be a metric space and $x \in X$.

Define $\nu(x) = \sup\{\varepsilon > 0 : B(x,\varepsilon) \text{ is compact}\}$, if x has some compact neighbourhood. Otherwise set $\nu(x) = 0$.

The set $\{x : \nu(x) = 0\}$ of all points of non-local compactness of X will be denoted by $nlc(X)$. E.g. $nlc(\mathbb{R}) = \emptyset$.

The next result gives a sequential characterization for cofinally complete metric spaces.

Theorem 23. [2] A metric space X is cofinally complete if and only if every sequence (x_n) in X with $\lim\limits_{n\to\infty} \nu(x_n) = 0$ has a convergenet subsequence.

In what follows, we characterize cofinally complete metric spaces using continuous functions. We begin with the following defintion.

Definition 19. Let X,Y be metric spaces and $f : X \to Y$ be a function. Then f is *uniformly locally bounded* if for some $\delta > 0$, $f(B(x,\delta))$ is bounded, $\forall\, x \in X$.

Theorem 24. [2] Let (X,d) be a metric space. Then the following conditions are equivalent:
a) X is a cofinally complete metric space.
b) If $f : X \to Y$ (Y being any metric space) is continuous. then f is uniformly locally bounded.
c) If $f : X \to \mathbb{R}$ is continuous, then f is uniformly locally bounded.

We have seen that the class of UC metric spaces is properly contained in the class of cofinally complete metric spaces. Since every continuous function on a UC metric space is uniformly continuous, we can expect a similar result for cofinally complete metric spaces by involving a proper subclass of continuous functions. The following theorem meets the expectation.

Theorem 25. [2] Let (X,d) be a metric space. Then the following conditions are equivalent:
a) X is a cofinally complete metric space.
b) $f : X \to Y$ (Y being any metric space) is continuous on X such that for $\varepsilon > 0$, f is uniformly continuous on $\{x : \nu(x) > \varepsilon\} \implies f$ is uniformly continuous on X.
c) If $f : X \to \mathbb{R}$ is bounded and continuous on X such that for $\varepsilon > 0$, f is uniformly continuous on $\{x : \nu(x) > \varepsilon\} \implies f$ is uniformly continuous on X.

Alike UC and Bourbaki complete metric spaces, cofinally complete metric spaces can be characterized using measures of noncompactness functional.

Definition 20. For a metric space X and $E \subset X$, set $\overline{\nu}(E) = \sup\{\nu(x) : x \in E\}$ and $\underline{\nu}(E) = \inf\{\nu(x) : x \in E\}$.

Theorem 26. [2] Let (X,d) be a metric space. Then the following conditions are equivalent:
a) X is cofinally complete metric space.
b) For a nested sequence (F_n) of nonempty closed sets in X, $\lim_{n\to\infty} \underline{\nu}(F_n) = 0 \implies \bigcap_{n=1}^{\infty} F_n \neq \emptyset$,
c) For a nested sequence (F_n) of nonempty closed sets in X, $\lim_{n\to\infty} \overline{\nu}(F_n) = 0 \implies \bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

The characterization of total boundedness using Cauchy sequences is well-known [19]. The following result shows that total boundedness can also be characterized using cofinally Cauchy sequences.

Theorem 27. [2] A metric space X is totally bounded if and only if every sequence in X is cofinally Cauchy.

8. Cofinally Bourbaki complete metric spaces

Cofinally Cauchy sequences have been introduced by replacing residuality of the indices with that of cofinality in the definition of Cauchy sequences. In this section, we discuss another class of sequences that employ cofinality. They are called cofinally Bourbaki-Cauchy sequences. It can be shown that the class of related complete metric spaces (*where all such sequences cluster*) is intermediate between that of compact and complete metric spaces.

Definition 21. [10] A sequence (x_n) in a metric space (X,d) is called *cofinally Bourbaki-Cauchy* if for $\varepsilon > 0$ $\exists\, m \in \mathbb{N}, p \in X$ and an infinite subset N_ε of $\mathbb{N}$ such that $x_n \in B_\varepsilon^m(p)$, $\forall\, n \in \mathbb{N}_\varepsilon$.

Alike Bourbaki-Cauchy sequences, Bourbaki-bounded subsets can also be characterized using cofinally Bourbaki-Cauchy sequences:

Theorem 28. [10] Let (X,d) be a metric space and $B \subset X$. Then the following conditions are equivalent:

a) B is Bourbaki-bounded subset of X.
b) Every sequence in B is cofinally Bourbaki-Cauchy in X.

Definition 22. [10] A metric space X is called *cofinally Bourbaki complete* if every cofinally Bourbaki-Cauchy sequence in X has a convergent subsequence.

Clearly the class of cofinally Bourbaki complete metric spaces lies in between the classes of compact and complete metric spaces.

We note that cofinally Bourbaki complete metric spaces are cofinally complete. The converse is not true. Consider the following example.

Example 5. [10] Let $l_2 = \left\{(x_n) : (x_n) \text{ is a real sequence and } \sum_{n=1}^{\infty} |x_n|^2 < \infty\right\}$ be endowed with the 2-norm. Set $X = \bigcup_{n\in\mathbb{N}} A_n$, where

$$A_n = \{e_n\} \bigcup \left\{e_n + \frac{1}{n} e_k : k \in \mathbb{N}\right\},$$

$\forall\, n \in \mathbb{N}$. Then X is a discrete subspace of l_2 that fails to be cofinally Bourbaki complete. However it is Bourbaki complete since every sequence in X is eventually constant.

Theorem 29. [10] Every UC metric space is cofinally Bourbaki complete.

The converse is not true. For example, consider $\mathbb{R}$ with the usual metric. Thus the class of UC metric spaces is strictly smaller than that of cofinally Bourbaki complete metric spaces.

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